

# Darboux-Egorov system, bi-flat $F$ -manifolds and Painlevé VI

Paolo Lorenzoni

Dipartimento di Matematica e Applicazioni

Università di Milano-Bicocca, Via Roberto Cozzi 53, I-20125 Milano, Italy

paolo.lorenzoni@unimib.it

## Abstract

This is a generalization of the procedure presented in [3] to construct semisimple bi-flat  $F$ -manifolds  $(M, \nabla^{(1)}, \nabla^{(2)}, \circ, *, e, E)$  starting from homogeneous solutions of degree  $-1$  of Darboux-Egorov-system. The Lamé coefficients  $H_i$  involved in the construction are still homogeneous functions of a certain degree  $d_i$  but we consider the general case  $d_i \neq d_j$ . As a consequence the rotation coefficients  $\beta_{ij}$  are homogeneous functions of degree  $d_i - d_j - 1$ . It turns out that any semisimple bi-flat  $F$  manifold satisfying a natural additional assumption can be obtained in this way. Finally we show that three dimensional semisimple bi-flat  $F$ -manifolds are parametrized by solutions of the full family of Painlevé VI.

## 1 Introduction

A *bi-flat* semisimple  $F$ -manifold  $(M, \nabla^{(1)}, \nabla^{(2)}, \circ, *, e, E)$  is a manifold  $M$  endowed with a pair of flat connections  $\nabla^{(1)}$  and  $\nabla^{(2)}$ , a pair of products  $\circ$  and  $*$  on the tangent spaces  $T_u M$  and a pair of vector fields  $e$  and  $E$  satisfying the following conditions:

- the product  $\circ$  is commutative, associative and with unity  $e$ . Moreover it is semisimple; this means that there exists a special set of coordinates, called canonical coordinates, such that the structure constants of  $\circ$  reduce to the standard form  $c_{jk}^i = \delta_j^i \delta_k^i$ .
- the product  $*$  is also commutative, associative and with unity  $E$ . Moreover the operator  $L = E \circ$  has vanishing Nijenhuis torsion and functionally independent

eigenvalues. As a consequence, in canonical coordinates for  $\circ$ , the structure constants of  $*$  read  $c_{jk}^{*i} = \frac{1}{E^i(u^i)} \delta_j^i \delta_k^i$ .

- $\nabla^{(1)}$  is compatible with the product  $\circ$  and  $\nabla^{(2)}$  is compatible with the product  $*$ :

$$\nabla_l^{(1)} c_{jk}^i = \nabla_j^{(1)} c_{lk}^i, \quad \nabla_l^{(2)} c_{jk}^{*i} = \nabla_j^{(2)} c_{lk}^{*i} \quad (1.1)$$

- $\nabla^{(1)} e = 0$  and  $\nabla^{(2)} E = 0$ ,
- $\nabla^{(1)}$  and  $\nabla^{(2)}$  are almost hydrodynamically equivalent i.e.

$$(d_{\nabla^{(1)}} - d_{\nabla^{(2)}})(X \circ) = 0, \quad \text{or} \quad (d_{\nabla^{(1)}} - d_{\nabla^{(2)}})(X *) = 0 \quad (1.2)$$

for every vector fields  $X$ ; here  $d_{\nabla}$  is the exterior covariant derivative constructed from a connection  $\nabla$ .

Bi-flat  $F$ -manifolds are a natural generalization of Frobenius manifolds. In the Frobenius case  $\nabla^{(1)}$  is the Levi-Civita connection of a metric  $\eta$  which is invariant with respect to the product. This extra assumption has two important consequences:  
- in flat coordinates for  $\nabla^{(1)}$ , one has

$$\eta_{il} c_{jk}^l = \partial_i \partial_j \partial_k F$$

for a suitable function  $F$ , called *the Frobenius potential*.

- the associated integrable hierarchy of PDEs, the *principal hierarchy*, is Hamiltonian with respect to the local Poisson bracket of hydrodynamic type defined by the metric  $\eta$ .

This means that, in general, the structure constants of bi-flat  $F$  manifolds do not admit any Frobenius potential and the associated integrable hierarchies are not Hamiltonian with respect to a local Poisson bracket of hydrodynamic type, at least in the usual sense (they become Hamiltonian in a weaker sense if one considers local Poisson bracket on 1-forms [4]).

In [3] it was shown how to construct semisimple bi-flat  $F$ -manifolds starting from the solutions of the Darboux-Egorov system [7, 9]

$$\partial_k \beta_{ij} = \beta_{ik} \beta_{kj}, \quad k \neq i \neq j \neq k \quad (1.3)$$

$$e(\beta_{ij}) = 0, \quad (1.4)$$

$$(1.5)$$

augmented with the condition

$$E(\beta_{ij}) = -\beta_{ij}. \quad (1.6)$$

In the symmetric case  $\beta_{ij} = \beta_{ji}$  the construction reduces to the usual Dubrovin procedure to define semisimple Frobenius manifolds from solutions of Darboux-Egorov

system. The non trivial point in the generalization is the relation between the connection  $\nabla^{(1)}$  and the Lamé coefficients  $H_i$  involved in the construction: in the non symmetric case the connection  $\nabla^{(1)}$  is no longer the Levi-Civita connection of the diagonal metric  $\eta_{ii} = H_i^2$ .

In the present paper we further extend Dubrovin procedure considering instead of (1.6) the more general condition

$$E(\beta_{ij}) = (d_i - d_j - 1)\beta_{ij}. \quad (1.7)$$

This adds  $n - 1$  free parameters to the theory. Remarkably, in the case  $n = 3$  the system (1.3,1.4,1.7) is equivalent to the full family of Painlevé VI (a more precise statement will be given in Section 5). Notice that the additional constraint (1.7) is not compatible with  $\beta_{ij} = \beta_{ji}$  since

$$E(\beta_{ij}) - E(\beta_{ji}) = 2(d_i - d_j)\beta_{ij}$$

and therefore the case  $d_i \neq d_j$  does not produce new examples of Frobenius manifolds.

The paper is organized as follows. In Section 2 we show how to construct bi-flat  $F$  manifolds starting from solutions of (1.3,1.4,1.7). We also show that if we assume that the eigenvalues of  $E \circ$  are canonical coordinates, then all bi-flat  $F$  manifolds can be obtained in this way. The case  $n = 2$  and  $n = 3$  are treated in Section 3 and 4. Section 4 is also devoted to discuss how the solutions of the system (1.3,1.4,1.7) are related to the sigma form of Painlevé VI. In the final Section 5 we discuss an example.

## 2 From Darboux-Egorov system to bi-flat $F$ manifolds

From now on we will work in canonical coordinates  $(u^1, \dots, u^n)$  and we will denote by  $\partial_i$  the partial derivative  $\frac{\partial}{\partial u^i}$ . Moreover by definition  $e = \sum_{i=1}^n \partial_i$  and  $E = \sum_{i=1}^n u^i \partial_i$ .

**Theorem 2.1** *Let  $\beta_{ij}$  be a solution of the system (1.3,1.4,1.7) and  $(H_1, \dots, H_n)$  a solution of the system*

$$\partial_j H_i = \beta_{ij} H_j, \quad i \neq j \quad (2.1)$$

$$e(H_i) = 0, \quad (2.2)$$

*satisfying the condition*

$$E(H_i) = d_i H_i, \quad (2.3)$$

*then*

- the natural connection  $\nabla_1$  defined by

$$\begin{aligned}
\Gamma_{jk}^i &:= 0 & \forall i \neq j \neq k \neq i \\
\Gamma_{jj}^i &:= -\Gamma_{ij}^i & i \neq j \\
\Gamma_{ij}^i &:= \frac{H_j}{H_i} \beta_{ij} & i \neq j \\
\Gamma_{ii}^i &:= -\sum_{l \neq i} \Gamma_{li}^i,
\end{aligned} \tag{2.4}$$

- the dual connection  $\nabla_2$  defined by

$$\begin{aligned}
\Gamma_{jk}^i &:= 0 & \forall i \neq j \neq k \neq i \\
\Gamma_{jj}^i &:= -\frac{u^i}{u^j} \Gamma_{ij}^i & i \neq j \\
\Gamma_{ij}^i &:= \frac{H_j}{H_i} \beta_{ij} & i \neq j \\
\Gamma_{ii}^i &:= -\sum_{l \neq i} \frac{u^l}{u^i} \Gamma_{li}^i - \frac{1}{u^i},
\end{aligned} \tag{2.5}$$

- the structure constants defined in the coordinates  $(u^1, \dots, u^n)$  by  $c_{jk}^i = \delta_j^i \delta_k^i$ ,
- the structure constants defined in the coordinates  $(u^1, \dots, u^n)$  by  $c_{jk}^{*i} = \frac{1}{u^i} \delta_j^i \delta_k^i$ ,
- the vector fields  $e$  and  $E$ ,

define a bi-flat semisimple  $F$ -manifold  $(M, \nabla_1, \nabla_2, \circ, *, e, E)$ .

*Proof.* The flatness of the connections  $\nabla^{(1)}$  and  $\nabla^{(2)}$  can be proved by straightforward computation. Moreover, by construction, the connection  $\nabla_1$  defined in (2.4) is compatible with the product  $c_{jk}^i = \delta_j^i \delta_k^i$  and satisfies  $\nabla_1 e = 0$  and the connection  $\nabla_2$  defined in (2.5) is compatible with the product  $c_{jk}^{*i} = \frac{\delta_j^i \delta_k^i}{u^i}$  and satisfies  $\nabla_2 E = 0$ .

Finally, the natural connection and the dual connection associated to the same functions  $H_i$  are almost hydrodynamically equivalent by definition since

$$\Gamma_{ij}^{(1)i} = \Gamma_{ij}^{(2)i} = \frac{H_j}{H_i} \beta_{ij}.$$

■

A natural question arises: does any bi-flat  $F$ -manifold come from a solution of the system (1.3,1.4,1.7,2.1,2.2,2.3)? The answer is given by the following theorem.

**Theorem 2.2** *Let  $(M, \nabla^{(1)}, \nabla^{(2)}, \circ, *, e, E)$  be a bi-flat  $F$ -manifold such that the eigenvalues of  $E \circ$  are canonical coordinates. Then there exist  $(H_i, \beta_{ij})$  satisfying the system (1.3,1.4,1.7,2.1,2.2,2.3) such that, in canonical coordinates*

$$\Gamma_{ij}^{(1)i} = \Gamma_{ij}^{(2)i} = \frac{H_j}{H_i} \beta_{ij}.$$

*Proof:* In canonical coordinates  $\nabla^{(1)}$  is given by (2.4) and  $e = \sum_l \frac{\partial}{\partial u^l}$ . Moreover, due to the additional assumption in canonical coordinates  $E = \sum_l u^l \frac{\partial}{\partial u^l}$  and  $\nabla^{(2)}$  is given by (2.5). Since  $\nabla^{(1)}$  and  $\nabla^{(2)}$  are almost hydrodynamically equivalent we have also

$$\Gamma_{ij}^{(1)i} = \Gamma_{ij}^{(2)i} := \Gamma_{ij}^i, \quad \forall i \neq j.$$

Now we have to exploit the flatness of  $\nabla^{(1)}$  and  $\nabla^{(2)}$ . From

$$R_{ikj}^{(1)i} = R_{ikj}^{(2)i} = \partial_k \Gamma_{ij}^i - \partial_j \Gamma_{ik}^i = 0,$$

it follows that there exist  $H_i$  such that

$$\Gamma_{ij}^i = \partial_j \ln H_i$$

Clearly  $H_i$  is defined up to a multiplicative factor depending only on  $u^i$ . Using  $R_{iji}^{(1)i} = 0$  and  $R_{ijl}^{(1)i} = 0$  we obtain

$$e(\Gamma_{ij}^i) = \partial_i \Gamma_{ij}^i + \sum_{l \neq i} \partial_l \Gamma_{ij}^i = \partial_j \Gamma_{ii}^{(1)i} + \sum_{l \neq i} \partial_l \Gamma_{ij}^i = - \sum_{l \neq i} \partial_j \Gamma_{il}^i + \sum_{l \neq i} \partial_j \Gamma_{il}^i = 0.$$

This implies  $\partial_j \left( \frac{e(H_i)}{H_i} \right) = 0$ , that is  $e(H_i) = c_i(u^i) H_i$ . Due to the freedom in the choice of  $H_i$ , without loss of generality we can assume  $c^i = 0$ . Similarly, using the flatness of the dual connection (in particular  $R_{iji}^{(2)i} = 0$  and  $R_{ijl}^{(2)i} = 0$ ) we obtain

$$\begin{aligned} E(\Gamma_{ij}^i) &= u^i \partial_i \Gamma_{ij}^i + \sum_{l \neq i} u^l \partial_l \Gamma_{ij}^i = u^i \partial_j \Gamma_{ii}^{(2)i} + \sum_{l \neq i} u^l \partial_l \Gamma_{ij}^i = \\ &= - \sum_{l \neq i} \partial_j (u^l \Gamma_{il}^i) + \sum_{l \neq i} u^l \partial_j \Gamma_{il}^i = - \Gamma_{ij}^i \end{aligned}$$

and, as a consequence:

$$\partial_j (E(\ln H_i)) = E(\partial_j \ln H_i) + \partial_j \ln H_i = 0, \quad \forall j \neq i.$$

This means that  $E(H_i) = d_i(u^i) H_i$ . We have to prove  $\partial_i d_i = 0$ . By straightforward computation we obtain

$$\begin{aligned} \partial_i d_i &= \partial_i \left( \frac{E(H_i)}{H_i} \right) = \frac{E(\partial_i H_i) + \partial_i H_i}{H_i} - \frac{E(H_i) \partial_i H_i}{H_i^2} = \\ &= \frac{E \left( - \sum_{l \neq i} \partial_l H_i \right) - \sum_{l \neq i} \partial_l H_i + d^i \sum_{l \neq i} \partial_l H_i}{H_i} = \\ &= \frac{- \sum_{l \neq i} \partial_l (E(H_i)) + d_i \sum_{l \neq i} \partial_l H_i}{H_i} = 0. \end{aligned}$$

Let us define the rotation coefficients as

$$\beta_{ij} = \frac{\partial_j H_i}{H_j} = \frac{H_i}{H_j} \Gamma_{ij}^i.$$

It remains to prove (1.4), (1.7) and (1.3). Due to  $e(\beta_{ij}) = 0$ ,  $E(H_i) = d_i H_i$  and  $E(\Gamma_{ij}^i) = -\Gamma_{ij}^i$ , the first and the second ones are elementary. The last one follows from  $R_{jki}^{(1)i} = R_{jki}^{(2)i} = 0$ :

$$\begin{aligned}
0 &= \partial_k \Gamma_{ij}^i + \Gamma_{ik}^i \Gamma_{ij}^i - \Gamma_{ij}^i \Gamma_{jk}^j - \Gamma_{ik}^i \Gamma_{kj}^k = \\
&\partial_k \left( \frac{H_j}{H_i} \beta_{ij} \right) + \frac{H_k H_j}{(H_i)^2} \beta_{ik} \beta_{ij} - \frac{H_k}{H_i} \beta_{ij} \beta_{jk} - \frac{H_j}{H_i} \beta_{ik} \beta_{kj} = \\
&\frac{\partial_k H_j}{H_i} \beta_{ij} - \frac{H_j \partial_k H_i}{(H_i)^2} \beta_{ij} + \frac{H_j}{H_i} \partial_k \beta_{ij} + \frac{H_k H_j}{(H_i)^2} \beta_{ik} \beta_{ij} - \frac{H_k}{H_i} \beta_{ij} \beta_{jk} - \frac{H_j}{H_i} \beta_{ik} \beta_{kj} = \\
&\frac{H_j}{H_i} (\partial_k \beta_{ij} - \beta_{ik} \beta_{kj}).
\end{aligned}$$

■

**Remark 2.3** Both the systems (1.3,1.4,1.7) and (2.1,2.2) (given  $\beta_{ij}$  satisfying (1.3,1.4)) are compatible. The proof is a straightforward (not short) computation. For arbitrary values of the constant  $d$ , system (2.1,2.2,2.3) does not admit solutions. The choice of the right degrees of homogeneity can be done adapting the procedure used by Dubrovin in [8] for the symmetric case.

The key observation is that the system (1.3,1.4,1.7) can be written in the Lax form <sup>1</sup>

$$\partial_k V = [V, W]$$

where  $V_{ij} = (u^j - u^i) \beta_{ij} - (d_j - d_1) \delta_j^i$  and  $W_{ij} = \delta_i^k \beta_{kj} - \beta_{ik} \delta_j^k$  (clearly instead of  $d_1$  we can choose  $d_2, \dots, d_n$ ). Moreover the system (2.1,2.2) is equivalent to

$$\partial_k H = -WH$$

where  $H = (H_1, \dots, H_n)$ . Using these facts it is easy to check that

- the matrix  $V$  acts on the space of solutions of the linear system (2.1,2.2),
- the eigenvalues of  $V$  do not depend on  $u$ .
- $d_1$  must be an eigenvalue of  $V$ . Indeed the eigenvectors  $H^{(\alpha)} = (H_1^{(\alpha)}, \dots, H_n^{(\alpha)})$  of  $V$  satisfy the equation:

$$E(H_i^{(\alpha)}) = (d_i - d_1 + \mu) H_i^{(\alpha)}.$$

---

<sup>1</sup>by definition  $\beta_{ii} = 0$ .

### 3 Examples in the case $n = 2$

In this case the Egorov-Darboux system reduces to

$$\begin{aligned}\frac{\partial \beta_{ij}}{\partial u^1} + \frac{\partial \beta_{ij}}{\partial u^2} &= 0, \\ u^1 \frac{\partial \beta_{ij}}{\partial u^1} + u^2 \frac{\partial \beta_{ij}}{\partial u^2} &= (d_i - d_j - 1) \beta_{ij}.\end{aligned}$$

The first equations tell us that the rotation coefficients depend only on the difference  $(u^1 - u^2)$ . The remaining equations tell us that they are homogeneous functions of degree  $-1$ . This gives us

$$\begin{aligned}\beta_{12} &= C_1(u^1 - u^2)^{d_1 - d_2 - 1}, \\ \beta_{21} &= C_2(u^1 - u^2)^{d_2 - d_1 - 1}.\end{aligned}$$

To construct the natural connections we need to solve the system for the Lamé coefficients:

$$\begin{aligned}\frac{\partial H_i}{\partial u^1} + \frac{\partial H_i}{\partial u^2} &= 0, \\ u^1 \frac{\partial H_i}{\partial u^1} + u^2 \frac{\partial H_i}{\partial u^2} &= d_i H_i, \\ \partial_2 H_1 &= C_1(u^1 - u^2)^{d_1 - d_2 - 1} H_2, \\ \partial_1 H_2 &= C_2(u^1 - u^2)^{d_2 - d_1 - 1} H_1.\end{aligned}$$

The first two equations imply

$$\begin{aligned}H_1 &= D_1(u^1 - u^2)^{d_1}, \\ H_2 &= D_2(u^1 - u^2)^{d_2}.\end{aligned}$$

Due to the remaining equations the constants  $D_1, D_2, d_1, d_2$  obey two additional constraints:

$$-d_1 D_1 = C_1 D_2$$

and

$$d_2 D_2 = C_2 D_1.$$

Multiplying both equations we obtain

$$d_1 d_2 = -C_1 C_2. \tag{3.1}$$

The same result can be obtained computing the eigenvalues of the matrix  $V$

$$\begin{pmatrix} 0 & -C_1 \\ C_2 & d_1 - d_2 \end{pmatrix}. \tag{3.2}$$

We have

$$\lambda = \frac{d_1 - d_2 \pm \sqrt{(d_1 - d_2)^2 - 4C_1C_2}}{2}.$$

If we impose that  $d_1$  is an eigenvalue we obtain the constraint (3.1).

For any choice of  $C_1$  and  $C_2$  the natural and dual connections  $\nabla_1$  and  $\nabla_2$  are defined by (2.4) and (2.5) with

$$\begin{aligned}\Gamma_{12}^1 &= \Gamma_{12}^{(1)1} = \Gamma_{12}^{(1)} = \frac{D_2}{D_1} \frac{C_1}{u^1 - u^2} = \frac{d_1}{u^2 - u^1}, \\ \Gamma_{21}^2 &= \Gamma_{12}^{(2)1} = \Gamma_{12}^{(2)} = \frac{D_1}{D_2} \frac{C_2}{u^1 - u^2} = \frac{d_2}{u^1 - u^2}.\end{aligned}$$

## 4 Bi-flat $F$ -manifolds in dimension $n = 3$

In this Section we show that the system (1.3,1.4,1.7) is equivalent to the sigma form of Painlevé VI. In literature, the relation between Darboux-Egorov (or the related  $N$ -wave system) and Painlevé VI has been studied by several authors (for instance [10, 13, 12, 6, 1]). The proof we present here is elementary. In one direction (from Darboux-Egorov to Painlevé VI) it is based on [1]. In the other direction we extend the proof given in [3] in the case  $d_i = d_j$ .

First of all, we observe that, due to (1.4) and (1.7), the rotation coefficients  $\beta_{ij}$  are homogeneous functions of degree  $d_i - d_j - 1$  depending only on the difference of the coordinates. Without loss of generality we can write them in the form

$$\begin{aligned}\beta_{12} &= \frac{1}{u^2 - u^1} F_{12} \left( \frac{u^3 - u^1}{u^2 - u^1} \right) (u^2 - u^1)^{d_1 - d_2} \\ \beta_{21} &= \frac{1}{u^2 - u^1} F_{21} \left( \frac{u^3 - u^1}{u^2 - u^1} \right) (u^2 - u^1)^{d_2 - d_1} \\ \beta_{32} &= \frac{1}{u^3 - u^2} F_{32} \left( \frac{u^3 - u^1}{u^2 - u^1} \right) (u^2 - u^1)^{d_3 - d_2} \\ \beta_{23} &= \frac{1}{u^3 - u^2} F_{23} \left( \frac{u^3 - u^1}{u^2 - u^1} \right) (u^2 - u^1)^{d_2 - d_3} \\ \beta_{13} &= \frac{1}{u^3 - u^1} F_{13} \left( \frac{u^3 - u^1}{u^2 - u^1} \right) (u^2 - u^1)^{d_1 - d_3} \\ \beta_{31} &= \frac{1}{u^3 - u^1} F_{31} \left( \frac{u^3 - u^1}{u^2 - u^1} \right) (u^2 - u^1)^{d_3 - d_1}\end{aligned} \tag{4.1}$$



Putting (4.1) into the system (1.3) we obtain the system (4.2) for the functions  $F_{ij}$ .

$$\begin{aligned}
\frac{d}{dz}F_{12} &= \frac{1}{z(z-1)}F_{13}F_{32} \\
\frac{d}{dz}F_{13} &= -\frac{1}{z-1}F_{12}F_{23} + \frac{d_1-d_3}{z}F_{13} \\
\frac{d}{dz}F_{21} &= \frac{1}{z(z-1)}F_{23}F_{31} \\
\frac{d}{dz}F_{23} &= \frac{1}{z}F_{21}F_{13} + \frac{d_2-d_3}{z-1}F_{23} \\
\frac{d}{dz}F_{31} &= -\frac{1}{z-1}F_{32}F_{21} + \frac{d_3-d_1}{z}F_{31} \\
\frac{d}{dz}F_{32} &= \frac{1}{z}F_{31}F_{12} + \frac{d_3-d_2}{z-1}F_{32},
\end{aligned} \tag{4.2}$$

where the independent variable  $z := \frac{u^3-u^1}{u^2-u^1}$ .

Now we discuss how the non-autonomous systems of ODEs (4.2) for the  $F_{ij}$  can be reduced to the sigma form of Painlevé VI.

**Theorem 4.1** *System (4.2) is equivalent to the following equation:*

$$\begin{aligned}
&z^2(z-1)^2(f'')^2 + 4[f'(zf' - f)^2 - (f')^2(zf' - f)] - (2R^2 + d_{13}^2)(f')^2 - d_{21}^2(zf' - f)^2 + \\
&- 2d_{21}d_{13}f'(zf' - f) - ((d_{13} + d_{23})R^2 + 2D)d_{21}(zf' - f) + \\
&- [((d_{13} + d_{23})R^2 + 2D)d_{13} + R^4]f' - \left(D + \frac{(d_{13} + d_{23})R^2}{2}\right)^2 = 0
\end{aligned} \tag{4.3}$$

After the substitution  $f = \psi + az = \phi = az + b$  with  $a = \frac{d_{21}^2}{4}$  and  $b = -\frac{d_{21}d_{23}}{4}$  the equation (4.3) reduces to

$$\begin{aligned}
&z^2(z-1)^2(\phi'')^2 + 4[\phi'(z\phi' - \phi)^2 - (\phi')^2(z\phi' - \phi)] - [2R^2 + d_{13}^2 + d_{21}d_{23}](\phi')^2 + \\
&- \left[2Dd_{21} + (d_{13}d_{21} + d_{23}d_{21})R^2 + \frac{d_{21}^4}{4} + \frac{d_{21}^3d_{13}}{2}\right](z\phi' - \phi) + \\
&- \left[R^4 + 2Dd_{13} + ((d_{21}^2 + d_{13}^2 + d_{23}d_{13})R^2 - \frac{d_{21}^2d_{23}^2}{4} + \frac{d_{21}^2d_{13}d_{23}}{2} + \frac{d_{21}^3d_{23}}{2} + \frac{d_{21}^2d_{13}^2}{2})\right]\phi' + \\
&- D^2 - DR^2(d_{13} + d_{23}) - \frac{R^4}{4}[d_{21}^2 + (d_{13} + d_{23})^2] - \frac{D}{2}d_{21}^2(d_{13} + d_{23}) + \\
&- \frac{R^2}{8}d_{21}^2[d_{21}^2 + 4d_{13}d_{23} + 2d_{13}^2 + 2d_{23}^2] - \frac{d_{21}^4}{16}[d_{23}d_{21} + d_{13}^2 + 2d_{13}d_{23}]
\end{aligned} \tag{4.4}$$

which is the sigma form of Painlevé VI equation:

$$\begin{aligned}
&z^2(z-1)^2(\sigma'')^2 + 4[\sigma'(z\sigma' - \sigma)^2 - (\sigma')^2(z\sigma' - \sigma)] - 4v_1v_2v_3v_4(z\sigma' - \sigma) + \\
&- (\sigma')^2\left(\sum_{k=1}^4 v_k^2\right) - \sigma'\left(\sum_{i<j}^4 v_i^2v_j^2 - 2v_1v_2v_3v_4\right) - \sum_{i<j<k}^4 v_i^2v_j^2v_k^2.
\end{aligned} \tag{4.5}$$

where the parameters  $v_1^2, v_2^2, v_3^2, v_4^2$  are the roots of the polynomial

$$\begin{aligned}
& \lambda^4 - (2R^2 + d_{13}^2 - d_{21}d_{13})\lambda^3 + \\
& + \left[ R^4 + D(2d_{13} + d_{21}) + \left( \frac{d_{13}d_{21}}{2} + \frac{d_{23}d_{21}}{2} + d_{21}^2 + d_{13}^2 + d_{23}d_{13} \right) R^2 + \right. \\
& - \frac{d_{21}^2d_{23}^2}{4} + \frac{d_{21}^4}{8} + \frac{d_{21}^3d_{13}}{4} + \frac{d_{21}^2d_{13}d_{23}}{2} + \frac{d_{21}^3d_{23}}{2} + \frac{d_{21}^2d_{13}^2}{2} \left. \right] \lambda^2 + \\
& - \left[ D^2 + DR^2(d_{13} + d_{23}) - \frac{R^4}{4} (d_{21}^2 + (d_{13} + d_{23})^2) + \frac{D}{2} d_{21}^2 (d_{13} + d_{23}) + \right. \\
& + \frac{R^2}{8} d_{21}^2 (d_{21}^2 + 4d_{13}d_{23} + 2d_{13}^2 + 2d_{23}^2) + \frac{d_{21}^4}{16} (d_{23}d_{21} + d_{13}^2 + 2d_{13}d_{23}) \left. \right] \lambda + \\
& + \left[ \frac{D}{2} d_{21} + \frac{R^2}{4} (d_{13}d_{21} + d_{23}d_{21}) + \frac{d_{21}^4}{16} + \frac{d_{21}^3d_{13}}{8} \right]^2.
\end{aligned} \tag{4.6}$$

*Proof.* By straightforward computation we get

$$\frac{d}{dz}(F_{12}F_{21} + F_{13}F_{31} + F_{23}F_{32}) = 0$$

and

$$\frac{d}{dz}(F_{23}F_{31}F_{12} - F_{13}F_{32}F_{21} + d_{23}F_{13}F_{31} + d_{13}F_{23}F_{32}) = 0$$

where  $d_{ij} := d_i - d_j$ . This implies

$$F_{12}F_{21} + F_{13}F_{31} + F_{23}F_{32} = -R^2 \tag{4.7}$$

and

$$F_{23}F_{31}F_{12} - F_{13}F_{32}F_{21} + d_{23}F_{13}F_{31} + d_{13}F_{23}F_{32} = D \tag{4.8}$$

for some constants  $R$  and  $D$ .

Let us introduce a function  $f$  defined, up to a constant, by

$$F_{12}F_{21} := f' \tag{4.9}$$

Due to equations (4.2), we have

$$\begin{aligned}
& \frac{d}{dz}(F_{13}F_{31}) = F_{13}'F_{31} + F_{13}F_{31}' = \\
& = -\frac{1}{z-1}F_{12}F_{23}F_{31} + \frac{d_1 - d_3}{z}F_{13}F_{31} - \frac{1}{z-1}F_{32}F_{21}F_{13} + \frac{d_3 - d_1}{z}F_{13}F_{31} = \\
& = -z\frac{d}{dz}(F_{12}F_{21}) = F_{12}F_{21} - \frac{d}{dz}(zF_{12}F_{21}) = \frac{d}{dz}(f - zf')
\end{aligned}$$

Thus, choosing the integration constant equal to  $-\frac{R^2}{2}$  we have

$$F_{13}F_{31} = f - zf' - \frac{R^2}{2}. \tag{4.10}$$

and consequently

$$F_{23}F_{32} = -R^2 - F_{12}F_{21} - F_{13}F_{31} = (z-1)f' - f - \frac{R^2}{2}. \quad (4.11)$$

We want to derive a second order ODE for the function  $f$ . This can be easily done writing the second derivative of  $f$  in terms of the products  $F_{12}F_{21}$ ,  $F_{13}F_{31}$  and  $F_{23}F_{32}$ . We have

$$\begin{aligned} [z(z-1)f'']^2 &= \left[ z(z-1) \frac{d}{dz} (F_{12}F_{21}) \right]^2 = [F_{21}F_{13}F_{32} + F_{12}F_{31}F_{23}]^2 \\ &= 4(F_{12}F_{21}F_{13}F_{31}F_{23}F_{32}) + (D - d_{23}F_{13}F_{31} - d_{13}F_{23}F_{32})^2 = \\ &= 4f'g_1g_2 + [D - d_{23}g_1 - d_{13}g_2]^2, \end{aligned}$$

where  $g_1 = f - zf' - \frac{R^2}{2}$ ,  $g_2 = -f + (z-1)f' - \frac{R^2}{2}$ . Expanding the above expression, after some computations one obtains the equation (4.3).

This proves that given a solution of system (4.2) we can construct a solution of (4.3).

Viceversa given any solution  $f$  of (4.3) the corresponding solution  $F_{ij}$  of (4.2) is defined by

$$\begin{aligned} F_{12} &= \sqrt{f'} \exp \left( - \int_{z_0}^z \left[ \frac{\varphi}{2t(t-1)f'} \right] dt + C_{12} \right), \\ F_{21} &= \sqrt{f'} \exp \left( \int_{z_0}^z \left[ \frac{\varphi}{2t(t-1)f'} \right] dt + C_{21} \right), \\ F_{13} &= \sqrt{g_1} \exp \left( - \int_{z_0}^z \left[ \frac{\varphi}{2(t-1)g_1} - \frac{d_{13}}{t} \right] dt + C_{13} \right), \\ F_{31} &= \sqrt{g_1} \exp \left( \int_{z_0}^z \left[ \frac{\varphi}{2(t-1)g_1} - \frac{d_{13}}{t} \right] dt + C_{31} \right), \\ F_{23} &= \sqrt{g_2} \exp \left( - \int_{z_0}^z \left[ \frac{\varphi}{2tg_2} - \frac{d_{23}}{t-1} \right] dt + C_{23} \right), \\ F_{32} &= \sqrt{g_2} \exp \left( \int_{z_0}^z \left[ \frac{\varphi}{2tg_2} - \frac{d_{23}}{t-1} \right] dt + C_{32} \right), \end{aligned} \quad (4.12)$$

where  $\varphi = D - d_{23}g_1 - d_{13}g_2$  and  $C_{ij}$  are integration constants satisfying the linear system

$$\begin{aligned} -C_{12} + C_{13} + C_{32} - \ln(f''(z_0)z_0(z_0-1) - \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) &= 0 \\ -C_{21} + C_{23} + C_{31} - \ln(f''(z_0)z_0(z_0-1) + \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) &= 0 \\ -C_{13} + C_{12} + C_{23} - \ln(f''(z_0)z_0(z_0-1) + \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) &= 0 \\ -C_{31} + C_{32} + C_{21} - \ln(f''(z_0)z_0(z_0-1) - \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) &= 0 \\ -C_{23} + C_{21} + C_{13} - \ln(f''(z_0)z_0(z_0-1) - \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) &= 0 \\ -C_{32} + C_{31} + C_{12} - \ln(f''(z_0)z_0(z_0-1) + \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) &= 0 \end{aligned} \quad (4.13)$$

The proof is a generalization of the proof given in [3] in the case  $d_{ij} = 0$  ( $\varphi = D$ ). Substituting (4.12) in (4.2), after some computations we obtain

$$\begin{aligned} & -C_{12} + C_{13} + C_{32} - \ln(f''(z_0)z_0(z_0 - 1) - \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) + \\ & - \int_{z_0}^z \frac{d}{dt} \ln(t(t-1)f'' - \varphi) dt + \int_{z_0}^z \frac{d}{dt} \ln[2\sqrt{f'g_1g_2}] dt + \\ & + \int_{z_0}^z \varphi \frac{g_1g_2 + (t-1)f'g_1 - tf'g_2}{2t(t-1)f'g_1g_2} dt - \int_{z_0}^z \left[ \frac{d_{23}}{t-1} - \frac{d_{13}}{t} \right] dt = 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} & -C_{12} + C_{13} + C_{32} - \ln(f''(z_0)z_0(z_0 - 1) - \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) + \\ & - \int_{z_0}^z \frac{t(t-1)f''' + (2t-1)f'' - \varphi'}{t(t-1)f'' - \varphi} dt + \int_{z_0}^z \frac{(t(t-1)f'' + \varphi) \frac{d}{dt}[f'g_1g_2]}{2t(t-1)f'g_1g_2f''} dt + \\ & - \int_{z_0}^z \left[ \frac{d_{23}t - d_{13}(t-1)}{t(t-1)} \right] dt = 0 \end{aligned}$$

Using the equation (4.3) written in the form

$$f'g_1g_2 = \frac{1}{4}(z(z-1)f'' + \varphi)(z(z-1)f'' - \varphi)$$

and the equation obtained from (4.3) by differentiating with respect to  $z$ , we obtain

$$\begin{aligned} & -C_{12} + C_{13} + C_{32} - \ln(f''(z_0)z_0(z_0 - 1) - \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) + \\ & - \int_{z_0}^z \frac{t(t-1)f''' + (2t-1)f'' - \varphi'}{t(t-1)f'' - \varphi} dt + \int_{z_0}^z \frac{2 \frac{d}{dt}[f'g_1g_2]}{t(t-1)(t(t-1)f'' - \varphi)f''} dt + \\ & - \int_{z_0}^z \left[ \frac{\varphi'}{t(t-1)f''} \right] dt = \\ & -C_{12} + C_{13} + C_{32} - \ln(f''(z_0)z_0(z_0 - 1) - \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) + \\ & - \int_{z_0}^z \frac{2t^2(t-1)^2f''f''' - 2t(t-1)\varphi'f'' + \frac{d}{dt}[t^2(t-1)^2](f'')^2 - 4 \frac{d}{dt}[f'g_1g_2]}{2t(t-1)(t(t-1)f'' - \varphi)f''} dt + \\ & - \int_{z_0}^z \left[ \frac{\varphi'}{t(t-1)f''} \right] dt = \\ & -C_{12} + C_{13} + C_{32} - \ln(f''(z_0)z_0(z_0 - 1) - \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) + \\ & - \int_{z_0}^z \frac{-2t(t-1)\varphi'f'' + 2\varphi\varphi'}{2t(t-1)(t(t-1)f'' - \varphi)f''} dt - \int_{z_0}^z \left[ \frac{\varphi'}{t(t-1)f''} \right] dt = \\ & -C_{12} + C_{13} + C_{32} - \ln(f''(z_0)z_0(z_0 - 1) - \varphi(z_0)) + \ln(2\sqrt{f'(z_0)g_1(z_0)g_2(z_0)}) \end{aligned}$$

This proves that the first equation of the system (4.13) comes from the first equation of the system (4.2). The remaining equations can be obtained in the same way.

Finally, performing the substitution  $f = \psi + az = \phi = az + b$  with  $a = \frac{d_{21}^2}{4}$  and  $b = -\frac{d_{21}d_{23}}{4}$ , it is easy to check that the equation (4.3) reduces to (4.4). Comparing (4.4) with (4.5), we conclude that the equation for  $\phi$  and for  $\sigma$  coincide iff

$$\begin{aligned}
\sum_{k=1}^4 v_k^2 &= (2R^2 + d_{13}^2 - d_{21}d_{13}) \\
\sum_{i<j}^4 v_i^2 v_j^2 &= R^4 + D(2d_{13} + d_{21}) + \left[ \frac{d_{13}d_{21}}{2} + \frac{d_{23}d_{21}}{2} + d_{21}^2 + d_{13}^2 + d_{23}d_{13} \right] R^2 + \\
&\quad -\frac{d_{21}^2 d_{23}^2}{4} + \frac{d_{21}^4}{8} + \frac{d_{21}^3 d_{13}}{4} + \frac{d_{21}^2 d_{13} d_{23}}{2} + \frac{d_{21}^3 d_{23}}{2} + \frac{d_{21}^2 d_{13}^2}{2} \\
\sum_{i<j<k}^4 v_i^2 v_j^2 v_k^2 &= D^2 + DR^2(d_{13} + d_{23}) - \frac{R^4}{4} [d_{21}^2 + (d_{13} + d_{23})^2] + \frac{D}{2} d_{21}^2 (d_{13} + d_{23}) + \\
&\quad + \frac{R^2}{8} d_{21}^2 [d_{21}^2 + 4d_{13}d_{23} + 2d_{13}^2 + 2d_{23}^2] + \frac{d_{21}^4}{16} [d_{23}d_{21} + d_{13}^2 + 2d_{13}d_{23}] \\
(v_1 v_2 v_3 v_4)^2 &= \left[ \frac{D}{2} d_{21} + \frac{R^2}{4} (d_{13}d_{21} + d_{23}d_{21}) + \frac{d_{21}^4}{16} + \frac{d_{21}^3 d_{13}}{8} \right]^2
\end{aligned}$$

In other words,  $v_i^2$  are the roots of the polynomial (4.6). ■

## 5 The generalized $\epsilon$ -system

The rotation coefficients

$$\beta_{ij} = \frac{\prod_{l \neq j} (u^j - u^l)^{\epsilon_l}}{\prod_{l \neq i} (u^i - u^l)^{\epsilon_l}} \frac{\epsilon_j}{u^i - u^j} \quad (5.1)$$

and the Lamé coefficients

$$H_i = \frac{1}{\prod_{l \neq i} (u^i - u^l)^{\epsilon_l}} \quad (5.2)$$

are solutions of the system (1.3, 1.4, 1.7, 2.1, 2.2, 2.3) with  $d_i = -\sum_{l \neq i} \epsilon_l$ .

Thus the associated natural connection  $\nabla^{(1)}$

$$\begin{aligned}
\Gamma_{jk}^{(1)i} &= 0 \quad \forall i \neq j \neq k \neq i \\
\Gamma_{jj}^{(1)i} &= -\Gamma_{ij}^{(1)i} \quad i \neq j \\
\Gamma_{ij}^{(1)i} &= \frac{\epsilon_j}{u^i - u^j} \quad i \neq j \\
\Gamma_{ii}^{(1)i} &= -\sum_{l \neq i} \Gamma_{li}^{(1)i},
\end{aligned}$$

the associated dual connection  $\nabla^{(2)}$

$$\begin{aligned}\Gamma_{jk}^{(2)i} &= 0 & \forall i \neq j \neq k \neq i \\ \Gamma_{jj}^{(2)i} &= -\frac{u^i}{u^j} \Gamma_{ij}^{(2)i} & i \neq j \\ \Gamma_{ij}^{(2)i} &= \frac{\epsilon_j}{u^i - u^j} & i \neq j \\ \Gamma_{ii}^{(2)i} &= -\sum_{l \neq i} \frac{u^l}{u^i} \Gamma_{li}^{(2)i} - \frac{1}{u^i},\end{aligned}$$

the products  $c_{jk}^i = \delta_j^i \delta_k^i$  and  $c_{jk}^{*i} = \frac{1}{u^i} \delta_j^i \delta_k^i$ , the vector fields  $e = \sum_{k=1}^n \partial_k$  and  $E = \sum_{k=1}^n u^k \partial_k$  define a bi-flat semisimple  $F$ -manifold structure for any choice of  $\epsilon_1, \dots, \epsilon_n$ .

## 5.1 Flat coordinates of the natural connection

We have to find a basis of flat exact 1-forms  $\theta = \theta_i du^i$ , that is,  $n$  independent solutions of the linear system of PDEs

$$\begin{aligned}\partial_j \theta_i - \frac{\epsilon_j \theta_i - \epsilon_i \theta_j}{u^i - u^j} &= 0, & i = 1, \dots, n, j \neq i \\ \partial_i \theta_i + \sum_{k \neq i} \frac{\epsilon_i \theta_k - \epsilon_k \theta_i}{u^k - u^i} &= 0, & i = 1, \dots, n,\end{aligned}\tag{5.3}$$

which is equivalent to

$$\begin{aligned}\partial_j \theta_i - \frac{\epsilon_j \theta_i - \epsilon_i \theta_j}{u^i - u^j} &= 0, & i = 1, \dots, n, j \neq i \\ \sum_{k=1}^n \partial_k \theta_i &= 0, & i = 1, \dots, n.\end{aligned}\tag{5.4}$$

In particular, we have that

$$0 = \sum_{k=1}^n \partial_k \theta_i = \sum_{k=1}^n \partial_i \theta_k = \partial_i \left( \sum_{k=1}^n \theta_k \right),$$

showing that  $\sum_{k=1}^n \theta_k$  is constant if  $\theta = \theta_k du^k$  is flat.

A trivial solution of the system (5.4) is given by  $\theta_j = \epsilon_j$  for all  $j$ , corresponding to the flat 1-form  $\theta^{(1)} = \sum_{l=1}^n \epsilon_l du^l = df^1$ , where  $f^1 = \sum_{l=1}^n \epsilon_l u^l$ . The other flat coordinates can be chosen according to

**Proposition 5.1** *If  $\sum_l \epsilon_l \neq 1$ , there exist flat coordinates  $(f^1, f^2, \dots, f^n)$  such that  $f_\epsilon^p(u)$  is a homogeneous function of degree  $(1 - \sum_l \epsilon_l)$  for all  $p = 2, \dots, n$ . Moreover  $e(f^p) = 0$  for all  $p = 2, \dots, n$ .*

The proof works exactly as in the case  $\epsilon_i = \epsilon_j$  (see [16]).

For instance, in the case  $n = 3$  following the same procedure explained in [16, 5] one can easily check that

$$\begin{aligned}
f^1 &= \epsilon_1 u^1 + \epsilon_2 u^2 + \epsilon_3 u^3 \\
f^2 &= \text{hypergeom} \left( \left[ -\frac{1}{2} + \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 - \epsilon_3 - \frac{1}{2}\sqrt{1 - \epsilon_1 - \epsilon_2}\sqrt{-\epsilon_1 + 8\epsilon_3 - \epsilon_2 + 1}, \right. \right. \\
&\quad \left. \left. -\frac{1}{2} + \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 - \epsilon_3 + \frac{1}{2}\sqrt{1 - \epsilon_1 - \epsilon_2}\sqrt{-\epsilon_1 + 8\epsilon_3 - \epsilon_2 + 1} \right], [-3\epsilon_3 + \epsilon_1], 1 + z \right) + \\
f^3 &= (1 + z)^{1+3\epsilon_3-\epsilon_1} \text{hypergeom} \left( \left[ \frac{1}{2} - \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 + 2\epsilon_3 - \frac{1}{2}\sqrt{1 - \epsilon_1 - \epsilon_2}\sqrt{-\epsilon_1 + 8\epsilon_3 - \epsilon_2 + 1}, \right. \right. \\
&\quad \left. \left. + \frac{1}{2} - \frac{1}{2}\epsilon_1 + \frac{1}{2}\epsilon_2 + 2\epsilon_3 + \frac{1}{2}\sqrt{1 - \epsilon_1 - \epsilon_2}\sqrt{-\epsilon_1 + 8\epsilon_3 - \epsilon_2 + 1} \right], [2 + 3\epsilon_3 - \epsilon_1], 1 + z \right)
\end{aligned} \tag{5.5}$$

where  $z = \frac{u^3 - u^2}{u^2 - u^1}$ .

## 5.2 Principal hierarchy

Given an  $F$ -manifold with compatible *flat* connection one can construct a hierarchy of integrable quasilinear PDEs called *principal hierarchy* [17]. It is defined in the following way, which is a straightforward generalization of the original definition given by Dubrovin in the case of Frobenius manifolds [8].

First of all, one defines the so-called *primary flows*:

$$u_{t_{(p,0)}}^i = c_{jk}^i X_{(p,0)}^k u_x^j, \tag{5.6}$$

where  $(X_{(1,0)}, \dots, X_{(n,0)})$  is a basis of flat vector fields. Then, starting from these flows, one can define the “higher flows” of the hierarchy,

$$u_{t_{(p,\alpha)}}^i = c_{jk}^i X_{(p,\alpha)}^k u_x^j, \tag{5.7}$$

by means of the following recursive relations:

$$\nabla_j X_{(p,\alpha)}^i = c_{jk}^i X_{(p,\alpha-1)}^k. \tag{5.8}$$

In this section we will study the principal hierarchy associated with the bi-flat  $F$ -manifold defined above. One of the flows is the generalized  $\epsilon$ -system [19].

**The primary flows.** In order to define the primary flows we need a frame of flat

vector fields  $X = X^i \frac{\partial}{\partial u^i}$ , that is,  $n$  independent solutions of the linear system of PDEs

$$\begin{aligned} \partial_j X^i + \frac{\epsilon_j X^i - \epsilon_i X^j}{u^i - u^j} &= 0, & i = 1, \dots, n, j \neq i \\ \partial_i X^i - \sum_{k \neq i} \frac{\epsilon_i X^k - \epsilon_k X^i}{u^k - u^i} &= 0, & i = 1, \dots, n \end{aligned} \quad (5.9)$$

which is equivalent to

$$\partial_j X^i + \frac{\epsilon_j X^i - \epsilon_i X^j}{u^i - u^j} = 0, \quad i = 1, \dots, n, j \neq i \quad (5.10)$$

$$[e, X] = 0. \quad (5.11)$$

Comparing (5.9) with (5.3), one notices that the components  $X^i$  of a flat vector fields for  $(\epsilon_1, \dots, \epsilon_n)$  are given by the components of a flat 1-form for  $(-\epsilon_1, \dots, -\epsilon_n)$ .

**The higher flows.** In the case of generalized  $\epsilon$ -system, the system (5.8) is equivalent to the system

$$\partial_j X_{(p,\alpha)}^i + \epsilon \frac{X_{(p,\alpha)}^i - X_{(p,\alpha)}^j}{u^i - u^j} = 0, \quad i = 1, \dots, n, j \neq i \quad (5.12)$$

$$[e, X_{(p,\alpha)}] = X_{(p,\alpha-1)}. \quad (5.13)$$

Since locally  $X_{(p,\alpha)}^i = \partial_i K_{(p,\alpha)}$  (the functions  $K_{(p,\alpha)}$  are the coefficients of the deformed flat coordinates for the generalized  $\epsilon$ -system with  $\epsilon_i \rightarrow -\epsilon_i$ ) the system (5.12) can be written as

$$(u^i - u^j) \partial_j \partial_i K_{(p,\alpha)} + (\epsilon_j \partial_i K_{(p,\alpha)} - \epsilon_i \partial_j K_{(p,\alpha)}) = 0, \quad i = 1, \dots, n, j \neq i \quad (5.14)$$

or in compact form as

$$dd_L K_{(p,\alpha)} = dK_{(p,\alpha)} \wedge df^1,$$

where  $f^1 = \sum_l \epsilon_l u^l$  and  $d_L$  is the differential associated with the torsionless tensor field  $L_j^i = u^i \delta_j^i$  [11]. This is a crucial remark because (5.14) can be recursively solved by

$$dK_{(p,\alpha)} = d_L K_{(p,\alpha)} - K_{(p,\alpha)} df^1.$$

Using this fact, it is easy to check that —apart from some critical values of  $\epsilon_i$ — the functions  $K_{(p,\alpha)}$  obtained in this way (properly normalized) provide the solutions of the full system (5.12, 5.13).

**Proposition 5.2** *Suppose that  $\sum_l \epsilon_l \neq -1$  and let  $(f^1 = \sum_l \epsilon_l u^l, f^2, \dots, f^n)$  be the flat coordinates of the natural connection of the  $(-\epsilon_1, \dots, -\epsilon_n)$ -system described in Proposition 5.1. If  $K_{(p,\alpha)}$  are the functions defined recursively by*

$$K_{(p,0)} = f^p, \quad dK_{(p,\alpha+1)} = d_L K_{(p,\alpha)} - K_{(p,\alpha)} df^1, \quad \alpha \geq 0, \quad (5.15)$$



and

$$Y_{(p,\alpha)}^i = -\frac{1}{\epsilon_i} \partial_i K_{(p,\alpha)}, \quad \alpha \geq 0, \quad (5.16)$$

then the vector fields  $X_{(1,\alpha)} = \frac{1}{\prod_{j=1}^\alpha (j - \sum_l \epsilon_l)} Y_{(1,\alpha)}$  (for  $\sum \epsilon_l \neq j$  with  $j = 1, \dots, \alpha$ ) and  $X_{(p,\alpha)} = \frac{1}{\alpha!} Y_{(p,\alpha)}$ , for  $p = 2, \dots, n$ , satisfy the recursion relations (5.8).

Moreover the recursion relations (5.15) are algebraically solved by

$$K_{(1,\alpha)} = \frac{1}{\alpha + 1} \left[ \sum_{l=1}^n (u^l)^2 \partial_l K_{(1,\alpha-1)} - \left( \sum_{l=1}^n \epsilon_l u^l \right) K_{(1,\alpha-1)} \right] \quad (5.17)$$

and, for  $\alpha \neq -1 - \sum_l \epsilon_l$ , by

$$K_{(p,\alpha)} = \frac{1}{\alpha + 1 + \sum_l \epsilon_l} \left[ \sum_{l=1}^n (u^l)^2 \partial_l K_{(p,\alpha-1)} - \left( \sum_{l=1}^n \epsilon_l u^l \right) K_{(p,\alpha-1)} \right], \quad p = 2, \dots, n. \quad (5.18)$$

The proof works as in the case  $\epsilon_i = \epsilon_j$  which is treated with details in [16].

**Remark 5.3** The vector fields  $Y_{(p,\alpha)}$  (5.16) define the twisted Lenard-Magri chain [2] associated to the almost hydrodynamically connections  $\nabla^{(1)}$  and  $\nabla^{(3)}$ :

$$\Gamma_{jk}^{(3)i} = \Gamma_{jk}^{(2)i} + (1 - \sum_l \epsilon_l) c_{jk}^{*i} = \Gamma_{jk}^{(2)i} + (1 - \sum_l \epsilon_l) \frac{1}{u^i} \delta_j^i \delta_k^i.$$

This means that they satisfy the following recursive relations

$$d_{\nabla^{(1)}} Y_{(n,\alpha)} = d_{\nabla^{(3)}} (E \circ Y_{(n-1,\alpha)}),$$

as one can easily verify by straightforward computation. This means that the recursive procedure to construct integrable hierarchies based on the Frölicher-Nijnhuis theory [15, 14] is a particular case of the more general setting developed in [2].

**Remark 5.4** For generic values of  $\epsilon_1, \dots, \epsilon_n$  the principal hierarchy is not hamiltonian w.r.t. a local Poisson bracket of hydrodynamic type. However according to [4] any flow can be written as

$$u_t^i = P^{ij} \alpha_j$$

where  $\alpha$  is a non exact 1 form,

$$P^{ij} = g^{ij} \partial_x - g^{il} \Gamma_{lk}^j u_x^k$$

is the local Poisson bivector of hydrodynamic type associated to a flat metric  $g$  compatible with the natural connection:  $\nabla^{(1)} g = 0$ .

### 5.3 Reciprocal transformations

To conclude this Section we apply the results of [5] to the generalized  $\epsilon$ -system.

**Theorem 5.5** *Suppose  $\beta_{ij}$  satisfies system (1.3,1.4,1.7) and  $H_i$  satisfies the corresponding system (2.1,2.2). Assume that  $A$  is a homogeneous flat coordinate of degree  $k$  of the natural connection satisfying the condition  $e(A) = 0$ , then*

$$\tilde{\beta}_{ij} := \beta_{ij} - \frac{H_i}{H_j} \partial_j \ln(A), \quad i \neq j, \quad (5.19)$$

and

$$\tilde{H}_i := \frac{H_i}{A}, \quad (5.20)$$

satisfy systems (1.3,1.4,1.7) and (2.1,2.2) respectively, with  $d_i$  replaced by  $d_i - k$  in (2.3).

In the case  $d_i = d_j$  the proof was given in [5]. The general case is completely similar.

Since  $n - 1$  flat coordinates of the generalized  $\epsilon$ -system satisfy the hypothesis of the above theorem with  $k = 1 - \sum_l \epsilon_l$ , we have immediately the following corollary.

**Corollary 5.6** *Let  $\beta_{ij}$  be the rotation coefficients (5.1) and  $H_i$  the Lamé coefficients (5.2), then the new rotation coefficients (5.19) and the new Lamé coefficients (5.20) with  $A = f^k$ ,  $k = 2, \dots, n$  define a new solution of systems (1.3,1.4,1.7) and (2.1,2.2) with  $d_i$  replaced by  $d_i - 1 + \sum_l \epsilon_l$ .*

In other words, using the language of [5], the reciprocal  $F$ -manifold associated with any flat coordinates  $f^2, \dots, f^n$  is still a bi-flat  $F$ -manifold.

### Acknowledgments

I thank Alessandro Arsie for many fruitful discussions.

### References

- [1] H. Aratyn, J. van de Leur, *Solutions of the Painlevé VI equation from reduction of integrable hierarchy in a Grassmannian approach*. Int. Math. Res. Not. IMRN 2008.
- [2] A. Arsie and P. Lorenzoni *F-manifolds with eventual identities, bidifferential calculus and twisted Lenard-Magri chains*, to appear in Int. Math. Res. Not.

- [3] A. Arsie and P. Lorenzoni *From Darboux-Egorov system to bi-flat F-manifolds*, arXiv:1205.2468.
- [4] A. Arsie and P. Lorenzoni *Poisson bracket on 1-forms and evolutionary partial differential equations*, arXiv:1207.3042.
- [5] A. Arsie and P. Lorenzoni *Reciprocal F-manifolds*, arXiv:1207.5731.
- [6] R. Conte, A. M. Grundland and M. Musette, *A reduction of the resonant three-wave interaction to the generic sixth Painlevé equation*, J. Phys. A **39** (2006), no. 39, 12115–12127.
- [7] G. Darboux, *Leçons sur les systèmes orthogonaux et les courbes curvilignes*, Paris, 1897.
- [8] B.A. Dubrovin, *Geometry of 2D topological field theories*, in: Integrable Systems and Quantum Groups, Montecatini Terme, 1993. Editors: M. Francaviglia, S. Greco. Springer Lecture Notes in Math. **1620** (1996), pp. 120–348.
- [9] D. Th. Egorov, *Collected papers on differential geometry*, Nauka, Moscow (1970) (in Russian).
- [10] A.S. Fokas, R.A. Leo, L. Martina, and G. Soliani, Phys. Lett. A115 (1986) 329.
- [11] A. Frölicher, A. Nijenhuis, *Theory of vector-valued differential forms*, Proc. Ned. Acad. Wetensch. Ser. A **59** (1956), 338–359.
- [12] S. Kakei, S., T. Kikuchi, *The sixth Painlevé equation as similarity reduction of  $\mathfrak{gl}(3)$  hierarchy*, Letters in Mathematical Physics **79** (2007): 221–234.
- [13] A. V. Kitaev, *On similarity reductions of the three-wave resonant system to the Painlevé equations*, J. Phys. A: Math. Gen. 23 (1990), 3543–3553.
- [14] P. Lorenzoni, *Flat bidifferential ideals and semi-Hamiltonian PDEs*, J. Phys. A **39** (2006), no. 44, 13701–13715.
- [15] P. Lorenzoni, F. Magri, *A cohomological construction of integrable hierarchies of hydrodynamic type*, Int. Math. Res. Not. **2005**, no. 34, 2087–2100.
- [16] P. Lorenzoni, M. Pedroni, *Natural connections for semi-Hamiltonian systems: The case of the  $\epsilon$ -system*, Letters in Mathematical Physics, **97** (2011), no. 1, 85–108.
- [17] P. Lorenzoni, M. Pedroni, A. Raimondo, *F-manifolds and integrable systems of hydrodynamic type*, Archivum Mathematicum **47** (2011), 163–180.

- [18] Y. Manin, *F-manifolds with flat structure and Dubrovin's duality*, Adv. Math. **198** (2005), no. 1, 5–26.
- [19] M.V. Pavlov, *Integrable hydrodynamic chains*, J. Math. Phys. **44** (2003), no. 9, 4134–4156.
- [20] S.P. Tsarev, *The geometry of Hamiltonian systems of hydrodynamic type. The generalised hodograph transform*, USSR Izv. **37** (1991) 397–419.